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Exact Modules and Serial Rings

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Adapting Azumaya's exactness [3] to bimodules, Camillo, Fuller, and Haack [4] said that a bimodule that has a composition series whose composition factors are balanced is an *exact bimodule* and that a ring R is an *exact ring* in case the regular bimodule ${}_RR_R$ is exact. We call a one-sided module ${}_RM$ *exact* if the bimodule ${}_RM_{\mathrm{End}({}_RM)}$ is an exact bimodule. After observing some properties of exact modules, we show that the exactness for rings and reflexive modules is preserved under Morita duality, and that a basic exact artinian ring and the endomorphism ring of its minimal cogenerator have the same Loewy series. This lends support to Azumaya's conjecture [3] that exact artinian rings have self-duality. Our concluding theorem proves that every module over a serial ring is exact.

Throughout this paper, all rings have identity elements and all modules are unitary. A presentation of Morita duality can be found in Anderson and Fuller [1, Sects. 23, 24] and we freely use terminologies and notations of [1].

LEMMA 1. *Let ${}_RM_S$ be a bimodule. The following statements are equivalent*:

(1) *${}_RM$ is balanced and* $\mathrm{End}({}_RM)$ *consists of multiplication by elements of S*;

(2) *${}_RM_S$ is balanced*;

(3) *M_S is balanced and* $\mathrm{End}(M_S)$ *consists of multiplication by elements of R.*

Proof. (2)⇒(1) and (3). By [1, Proposition 4.14].

(1)⇒(2). Let $f\in\mathrm{End}(M_S)$. For any $\varphi\in\mathrm{End}({}_RM)$, there is an $s\in S$ with

$$\varphi(x)=xs \qquad \text{for all} \quad x\in M.$$

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Then

$$f(xg) = f(xs) = f(x)s = f(x)\,g,$$

i.e., $f \in \mathrm{End}(M_{\mathrm{End}({}_RM)})$. Now ${}_RM$ is balanced, so f is given by left multiplication by an element of R.

$(3) \Rightarrow (2)$. By a similar proof.

Using Lemma 1, we note that the exactness of modules is preserved under isomorphisms, direct summands, and arbitrary copies.

LEMMA 2. *Let ${}_RM$ be an exact module.*

(1) *If ${}_RM \cong {}_RN$, then ${}_RN$ is exact, too;*

(2) *any summand of ${}_RM$ is exact;*

(3) *${}_RM^{(I)}$ is exact for any non-empty set I.*

Proof. (1) follows from [1, Exercise 4.14] and (2) follows from [4, Lemma 1.4]. For (3), let $T = \mathrm{End}({}_RM)$ and $S = \mathrm{End}({}_RM^{(I)})$ and we denote the elements of S by $s = (t_{ij}) \in \mathbb{RFM}_I(T)$ (see [1, Exercise 6.21]), where all $t_{ij} \in T$, and $([x_j])s = [\sum_i (x_i) t_{ij}]$ for $[x_j] \in M^{(I)}$. Let

$$M = M_0 > M_1 > \cdots > M_{n-1} > M_n = 0$$

be a composition series for ${}_RM_T$, then clearly

$$M^{(I)} = M_0^{(I)} > M_1^{(I)} > \cdots > M_{n-1}^{(I)} > M_n^{(I)} = 0$$

is a composition series for ${}_R(M^{(I)})_S$.

Since ${}_R(M_{k-1}/M_k)$ is balanced, so is ${}_R(M_{k-1}/M_k)^{(I)} \cong {}_R(M_{k-1}^{(I)}/M_k^{(I)})$ by [1, Exercise 14.8]. Now using Lemma 1 and the exactness of ${}_RM_T$, we conclude that ${}_R(M^{(I)})_S$ is exact.

A left injective module ${}_RE$ and a right projective module P_R is said to form a *pair* (Fuller [5, 6]) over a semiprimary ring R in case $E(T(Re_1)), \ldots, E(T(Re_m))$ represent all the indecomposable direct summands of E where $e_1, \ldots, e_m$ is an orthogonal set of primitive idempotents such that $e_1R, \ldots, e_mR$ are (to within isomorphism) the indecomposable direct summands of P, and such an idempotent $e = \sum_{i=1}^m e_i$ is called a *basic idempotent* for E and P.

The next theorem gives a relation between a pair formed by a left injective module and a right projective module over a left artinian ring.

THEOREM 3. *If finitely cogenerated injective module ${}_RE$ and finitely generated projective module P_R form a pair over a left artinian ring R, then ${}_RE$ is exact if and only if P_R is exact.*

Proof. By [4, Theorem 1.3] we can assume that R is basic, and by Lemma 2 we may assume that ${}_RE = E(Re/Je)$ and $P_R = eR$ for some basic idempotent e for E and P, where $J = J(R)$. Let $S = \mathrm{End}({}_RE)$, which is a basic semiperfect ring. And moreover, $S = \mathrm{End}({}_RE) \cong \mathrm{End}({}_{eRe}eE)$ by [5, Lemma 2.1].

($\Leftarrow$) Let

$$ {}_{eRe}eR_R = eI_0 > eI_1 > \cdots > eI_{n-1} > eI_n = 0 $$

be a composition series, where each $I_i \leqslant {}_RR_R$.

By [4, Lemma 2.1], each eI_{i-1}/eI_i is simple on each side, so

$$ {}_{eRe}(eI_{i-1}/eI_i) \cong eRf_1/eJf_1 \qquad \text{and} \qquad (eI_{i-1}/eI_i)_R \cong f_2R/f_2J $$

for some primitive idempotents f_1 in eRe and f_2 in R. Using [5, Theorem 4.1], we have a composition series

$$ {}_RE_S = r_E(I_n) > r_E(I_{n-1}) > \cdots > r_E(I_1) > r_E(I_0) = 0 $$

and by [5, Lemma 2.1]

$$ r_E(I_i)/r_E(I_{i-1}) \cong \mathrm{Hom}_{eRe}(eI_{i-1}/eI_i, eE) $$

as R-S-bimodules. The right hand side is a simple S-module by [11, Theorem 2.1]. And by [1, Theorem 22.2], the functor

$$ \mathrm{Hom}_{f_1Rf_1/f_1Jf_1}(eI_{i-1}/eI_i, -) = \mathrm{Hom}_{eRe}(eI_{i-1}/eI_i, -) $$

gives an equivalence from the category of left f_1Rf_1/f_1Jf_1-modules onto the category of left f_2Rf_2/f_2Jf_2-modules. So

$$ \begin{aligned} r_E(I_i)/r_E(I_{i-1}) &\cong \mathrm{Hom}_{eRe}(eI_{i-1}/eI_i, eE) \\ &= \mathrm{Hom}_{eRe}(eI_{i-1}/eI_i, f_1Rf_1/f_1Jf_1) \\ &\cong Rf_2/Jf_2 \end{aligned} $$

as left R-modules. Hence $r_E(I_i)/r_E(I_{i-1})$ is a balanced bimodule by [4, Lemma 2.1], and ${}_RE_S$ is an exact bimodule.

($\Rightarrow$) Since ${}_{eRe}eE_S$ is exact by [4, Lemma 1.4], and hence by [5, Lemma 2.3], ${}_{eRe}eR_R \cong \mathrm{Hom}_S(E, eE)$, which is an exact bimodule by [4, Lemma 2.4].

Modifying Azumaya's proof of [3, Theorem 4], we have the following.

LEMMA 4. *Suppose that ${}_RE_S$ defines a duality and T is a semilocal ring. If ${}_RM_T$ is an exact bimodule, then so is ${}_T\mathrm{Hom}_R(M, E)_S$.*

Proof. Let

$$_RM_T = M_0 > M_1 > \cdots > M_{n-1} > M_n = 0$$

be a composition series. By [4, Proposition 1.1], $_RM$ is finitely generated and hence E-reflexive. Let $_TM_S^* = \mathrm{Hom}_R(M, E)$, and by [1, Theorem 24.5], we have a composition series

$$_TM_S^* = r_{M^*}(M_n) > r_{M^*}(M_{n-1}) > \cdots > r_{M^*}(M_1) > r_{M^*}(M) = 0.$$

Consider the exact sequence of bimodules

$$0 \to {}_R(M_{i-1}/M_i)_T \to {}_R(M/M_i)_T \to {}_R(M/M_{i-1})_T \to 0.$$

Since $_RE$ is injective, we have an exact sequence of bimodules

$$0 \to {}_T(M/M_{i-1})_S^* \to {}_T(M/M_i)_S^* \to {}_T(M_{i-1}/M_i)_S^* \to 0,$$

where $(-)^* = \mathrm{Hom}_R(-, E)$. The second and the third terms of this sequence are naturally identified with $r_{M^*}(M_{i-1})$ and $r_{M^*}(M_i)$, respectively. Now since $_R(M_{i-1}/M_i)_T$ is simple balanced, $_R(M_{i-1}/M_i)$ is semisimple and

$$\bar{T} = T/r_T(M_{i-1}/M_i) \cong \mathrm{End}(_R(M_{i-1}/M_i))$$

is simple artinian. It follows that $_R(M_{i-1}/M_i)$ is a direct sum of a finite number of copies of $T(Re)$ for some primitive idempotent e in R. And since $_RE$ is a finitely cogenerated injective cogenerator, there is a positive integer n with

$$_T(r_{M^*}(M_i)/r_{M^*}(M_{i-1}))_S \cong (M_{i-1}/M_i)^* = \mathrm{Hom}_R(M_{i-1}/M_i, T(Re)^n)$$

which is simple balanced by [4, Lemma 2.4], because $_R(M_{i-1}/M_i)_T$ and $_R((Re)^n)_S$ are simple balanced.

The next two theorems show that the exactness for rings and reflexive modules are preserved under Morita duality.

THEOREM 5. *Suppose that $_RE_S$ defines a duality, and $_RM$ is exact and E-reflexive, then so is* $\mathrm{Hom}_R(M, E)_S$.

Proof. Let $M_S^* = \mathrm{Hom}_R(M, E)$, $T = \mathrm{End}(_RM)$, and $W = \mathrm{End}(M_S^*)$. Since R is semilocal, each composition factor of $_RM_T$ is semisimple, and hence has finite length as a left R-module because a semisimple E-reflexive module must have finite length by [1, Lemma 24.7]. Hence $_RM$ has finite length and T is semiprimary. So $_TM_S^*$ is exact by Lemma 4. The result follows from [1, Proposition 23.3].

In his thesis [7], Habeb proved that "(1) $\Rightarrow$ (2) $\Rightarrow$ (3)" of the following theorem.

THEOREM 6 [7, Theorem 2.19]. *Let R be a left artinian ring and ${}_RE$ a finitely cogenerated injective cogenerator. Then the following statements are equivalent*:

(1) *R is exact*;

(2) *${}_RE$ is exact*;

(3) $\mathrm{End}({}_RE)$ *is an exact artinian ring.*

Proof. Let $S = \mathrm{End}({}_RE)$.

(1) $\Leftrightarrow$ (2). See Theorem 3.

(1), (2) $\Rightarrow$ (3). Since ${}_RE_S$ is exact, ${}_SS_S = \mathrm{Hom}_R({}_RE_S, {}_RE_S)$ is exact by [4, Lemma 2.4]. According to [3], ${}_RE_S$ defines a duality, so S is exact right artinian, and hence S is also left artinian from Azumaya's observation that the exactness is left-right symmetric.

(3) $\Rightarrow$ (2). By [8, Theorem 2.3], ${}_RE_S$ defines a duality and hence ${}_RE_S \cong \mathrm{Hom}_S(S, E)$ is exact by Lemma 4.

In the following theorem, the equivalence of (1), (2), (3), and (4) is actually due to Camillo, Fuller, and Haack [4].

THEOREM 7. *Let R be a left artinian ring. The following statements are equivalent*:

(1) *R is exact*;

(2) *eR_R is exact for each primitive idempotent e in R*;

(3) *${}_RRe$ is exact for each primitive idempotent e in R*;

(4) *${}_{eRe}eRf_{fRf}$ is exact for all primitive idempotents e and f in R, where $eRf \neq 0$*;

(5) *${}_RE(T(Re))$ is exact for each primitive idempotent e in R.*

Proof. Assume that R is basic.

(2) $\Leftrightarrow$ (5). See Theorem 3.

(1) $\Rightarrow$ (2) and (3). See [4, Theorem 2.3].

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4). See [4, Lemma 1.4].

(4) $\Rightarrow$ (1). Let $J = J(R)$ and $\bar{R} = R/J^2$. Since ${}_{eRe}eRf_{fRf}$ is exact, ${}_{\bar{e}\bar{R}\bar{e}}\bar{e}\bar{R}\bar{f}_{\bar{f}\bar{R}\bar{f}}$ is exact, where $\bar{e}$ and $\bar{f}$ are idempotents in $\bar{R}$. Hence $\bar{R}$ is exact by [4, Proposition 2.7], and then R is exact by [4, Theorem 2.5].

The following three lemmas will be used very often in order to prove that a basic exact artinian ring and the endomorphism ring of its minimal cogenerator have the same upper and lower Loewy series.

LEMMA 8. *Let* ${}_RM_S$ *and* ${}_RE$ *be modules with* S *a semilocal ring. If* M_S *is semisimple, so is* ${}_S\mathrm{Hom}_R(M, E)$.

Proof. $MJ(S) = 0$ implies that $J(S)\,\mathrm{Hom}_R(M, E) = 0$.

From the proof of Lemma 4, the following result is immediate.

LEMMA 9. *If* ${}_RE_S$ *defines a duality and* $E_1 \leqslant E_2 \leqslant {}_RE_S$, *then* ${}_S(r_S(E_1)/r_S(E_2))_S \cong \mathrm{Hom}_R(E_2/E_1, E)$ *as bimodules.*

LEMMA 10. *Suppose that* ${}_RE_S$ *defines a duality. Let* $J = J(R)$, *and* $N = J(S)$. *Then*

(1) $r_E(J^i) = l_E(N^i)$ *for all* $i \geqslant 0$;

(2) $r_Er_R(J^i) = l_Er_S(N^i)$ *for all* $i \geqslant 0$;

(3) $r_El_R(J^i) = l_El_S(N^i)$ *for all* $i \geqslant 0$.

Proof. Use the induction on i, [1, Exercise 24.8], and Lemma 8.

From now on through Theorem 12, we make the following assumption

(#) R is a basic exact artinian ring with a basic set of primitive idempotents $e_1, ..., e_n$, ${}_RE = \bigoplus_{j=1}^n E(T(Re_j))$ the minimal injective cogenerator, and $S = \mathrm{End}({}_RE)$. Denote $J = J(R)$, $N = J(S)$, and $f_j\colon {}_RE \to E(T(Re_j))$ the projections.

Then ${}_RE_S$ defines a duality [3], and ${}_RE_S$ is an exact bimodule and S is a basic exact artinian ring by [7, Theorem 2.19] or Theorem 6. According to [3, Corollaries 3 and 5] and [4, Lemma 2.1], we have

$$c({}_RR) = c(R_R) = c({}_RE) = c(E_S) = c({}_SS) = c(S_S).$$

PROPOSITION 11. *If*

$${}_RR_R = I_0 > I_1 > \cdots > I_{n-1} > I_n = 0$$

is a composition series with

$${}_R(I_{i-1}/I_i) \cong T(Re_{j_i}) \qquad \text{and} \qquad (I_{i-1}/I_i)_R \cong T(e_{k_i}R),$$

then

$${}_SS_S = r_Sr_E(I_0) > r_Sr_E(I_1) > \cdots > r_Sr_E(I_{n-1}) > r_Sr_E(I_n) = 0$$

is a composition series with

$$_S(r_S r_E(I_{i-1})/r_S r_E(I_i)) \cong T(Sf_{j_i})$$

and

$$(r_S r_E(I_{i-1})/r_S r_E(I_i))_S \cong T(f_{k_i}S).$$

Proof. The exact bimodule $_R E_S$ has a composition series

$$_R E_S = r_E(I_n) > r_E(I_{n-1}) > \cdots > r_E(I_1) > r_E(I_0) = 0,$$

and then each $r_E(I_i)/r_E(I_{i-1})$ is simple on each side by [4, Lemma 2.1]. Moreover since

$$\begin{aligned} _R(r_E(I_i)/r_E(I_{i-1}))_S &\cong \mathrm{Hom}_R(I_{i-1}/I_i, E) \\ &\cong \mathrm{Hom}_R(I_{i-1}/I_i, E(T(Re_{j_i}))), \end{aligned}$$

we have $(r_E(I_i)/r_E(I_{i-1}))_S \cong T(f_{j_i}S)$, and $_R(r_E(I_i)/r_E(I_{i-1})) \cong T(Re_{k_i})$ since $e_{k_i}(r_E(I_i)/r_E(I_{i-1})) \neq 0$. Now we have a composition series

$$_S S_S = r_S r_E(I_0) > r_S r_E(I_1) > \cdots > r_S r_E(I_{n-1}) > r_S r_E(I_n) = 0,$$

and

$$\begin{aligned} _S(r_S r_E(I_{i-1})/r_S r_E(I_i))_S &\cong \mathrm{Hom}_R(r_E(I_i)/r_E(I_{i-1}), E) \\ &= \mathrm{Hom}_R(r_E(I_i)/r_E(I_{i-1}), E(T(Re_{k_i}))), \end{aligned}$$

and the results follow by the same arguments as above.

Two modules $_R M$ and $_S U$ are said to have the *same upper* (*lower*) *Loewy series* in case for each $i \geqslant 0$,

$$J^{i-1}M/J^iM \cong \bigoplus_j T(Re_j)^{k_{ij}}$$

if and only if

$$N^{i-1}U/N^iU \cong \bigoplus_j T(Sf_j)^{k_{ij}}$$

(resp.

$$r_M(J^i)/r_M(J^{i-1}) \cong \bigoplus_j T(Re_j)^{l_{ij}}$$

if and only if

$$r_U(N^i)/r_U(N^{i-1}) \cong \bigoplus_j T(Sf_j)^{l_{ij}}).$$

A similar definition stands for right modules.

Azumaya [3] conjectured that exact artinian rings have self-duality, i.e., $R \cong S$ in our notation (#). We are unable to settle Azumaya's conjecture here, but the next result gives further evidence.

THEOREM 12. *For each j, Re_j (resp. e_jR) and Sf_j (resp. f_jS) have the same upper and lower Loewy series. Consequently ${}_RR$ (resp. R_R) and ${}_SS$ (resp. S_S) have the same upper and lower Loewy series.*

Proof. We only prove the results for left modules. Let

$$R > \cdots > J^{i-1} = I_1 > I_2 > \cdots > I_{t-1} > I_t = J^i > \cdots > 0$$

be a composition series for ${}_RR_R$. Since each I_k/I_{k+1} is simple on each side by [4, Lemma 2.1],

$$J^{i-1}/J^i \cong \bigoplus_k I_k/I_{k+1}$$

as left and right modules. By [3, Theorem 2],

$$J^{i-1}e_j/J^ie_j \cong \bigoplus_k I_ke_j/I_{k+1}e_j$$

as left modules. Now by Proposition 11, we have a composition series

$${}_SS_S > \cdots > r_Sr_E(J^{i-1}) = r_Sr_E(I_1) > \cdots > r_Sr_E(I_t) = r_Sr_E(J^i) > \cdots > 0,$$

and using Lemma 10(1) we have

$$\begin{aligned} N^{i-1}/N^i &= r_Sl_E(N^{i-1})/r_Sl_E(N^i) = r_Sr_E(J^{i-1})/r_Sr_E(J^i) \\ &\cong \bigoplus_k (r_Sr_E(I_k)/r_Sr_E(I_{k+1})) \end{aligned}$$

as left and right modules. And then using [3, Theorem 2] again we have

$$N^{i-1}f_j/N^if_j \cong \bigoplus_k (r_Sr_E(I_k)f_j/r_Sr_E(I_{k+1})f_j)$$

as left modules. By Proposition 11 again

$$I_ke_j/I_{k+1}e_j \begin{cases} =0 \\ \cong T(Re_l) \end{cases}$$

if and only if

$$r_Sr_E(I_k)f_j/r_Sr_E(I_{k+1})f_j \begin{cases} =0 \\ \cong T(Sf_l). \end{cases}$$

The proof is completed for the upper Loewy series. For the lower Loewy series, let

$$R > \cdots > r_R(J^i) = I_1 > I_2 > \cdots > I_{t-1} > I_t = r_R(J^{i-1}) > \cdots > 0$$

be a composition series for ${}_R R_R$.

Since each I_k/I_{k+1} is simple on each side

$$r_R(J^i)/r_R(J^{i-1}) \cong \bigoplus_k I_k/I_{k+1}$$

as left modules. By [3, Theorem 2],

$$r_{Re_j}(J^i)/r_{Re_j}(J^{i-1}) = r_R(J^i)e_j/r_R(J^{i-1})e_j \cong \bigoplus_k I_k e_j/I_{k+1}e_j$$

as left modules. By Proposition 11, we have a composition series for ${}_S S_S$:

$$S > \cdots > r_S r_E r_R(J^i) = r_S r_E(I_1) > \cdots > r_S r_E(I_t) = r_S r_E r_R(J^{i-1}) > \cdots > 0.$$

Using Lemma 10(2), we have

$$\begin{aligned} r_S(N^i)/r_S(N^{i-1}) &= r_S r_E r_R(J^i)/r_S r_E r_R(J^{i-1}) \\ &\cong \bigoplus_k (r_S r_E(I_k)/r_S r_E(I_{k+1})) \end{aligned}$$

as left modules. By [3, Theorem 2] again,

$$\begin{aligned} r_{Sf_j}(N^i)/r_{Sf_j}(N^{i-1}) &= r_S(N^i)f_j/r_S(N^{i-1})f_j \\ &\cong r_S r_E(I_k)f_j/r_S r_E(I_{k+1})f_j \end{aligned}$$

as left modules. By Proposition 11 again, we get

$$I_k e_j/I_{k+1}e_j \begin{cases} = 0 \\ \cong T(Re_l) \end{cases}$$

if and only if

$$r_S r_E(I_k)f_j/r_S r_E(I_{k+1})f_j \begin{cases} = 0 \\ \cong T(Sf_l), \end{cases}$$

which completes the proof of the theorem.

Recall that a module is *uniserial* in case its submodules are linearly ordered by inclusion, and that an artinian ring is *serial* (generalized uniserial in the sense of Nakayama [10]) in case each of its indecomposable projective modules is uniserial. Nakayama [10, Theorem 17] proved that every module over a serial ring is a direct sum of uniserial modules, each of which is an epimorph of an indecomposable projective module. It follows that there are only finitely many non-isomorphic uniserial modules over a serial ring. Azumaya [3] proved that serial rings are exact rings.

Although we do not know if the exactness of modules is preserved by submodules and quotient modules, the next lemma can be obtained easily.

LEMMA 13. *Let ${}_RM$ be an exact module and $N \leqslant {}_RM_{\mathrm{End}({}_RM)}$. If ${}_RM$ is quasi-injective (resp. quasi-projective), then ${}_RN$ (resp. ${}_RM/N$) is exact.*

Let R be a serial ring and ${}_RM$ be a uniserial module. According to [10], ${}_RM \cong Re/Ke$ for some $K \leqslant {}_RR_R$ and primitive idempotent e in R. Now R is an exact ring, and hence ${}_RRe$ is an exact module by Theorem 7. It follows from the above lemma that ${}_RM$ is exact.

We need the following lemma to prove our concluding result.

LEMMA 14. *Let R be a ring, $M_1, ..., M_n$ be left uniserial R-modules of finite length with $M_i \not\cong M_j$ whenever $i \neq j$, and $c(M_1) \geqslant c(M_2) \geqslant \cdots \geqslant c(M_n)$. Let $M = \bigoplus_i M_i$ and $S = \mathrm{End}({}_RM)$. If*

$$N = N_1 \oplus \cdots \oplus N_k \oplus M_{k+1} \oplus \cdots \oplus M_n,$$

where each $N_i \leqslant {}_RM_i$ and

$$c(N_1) \geqslant c(N_2) \geqslant \cdots \geqslant c(N_k) \geqslant \max\{c(N_1) - 1, c(M_{k+1})\},$$

then $N \leqslant {}_RM_S$.

Proof. Denote the elements of S by $s = (s_{ij})$, where $s_{ij} \in \mathrm{Hom}_R(M_i, M_j)$. For $[m_j]_{j=1}^n \in M$ and $s = (s_{ij}) \in S$ we have $([m_j]_{j=1}^n)s = [\sum_{i=1}^n (m_i)s_{ij}]_{j=1}^n$. Let $N_i = M_i$ for $k+1 \leqslant i \leqslant n$. We need to show that

$$(N_i)s_{ij} \subseteq N_j$$

for all i and j, and $s_{ij} \in \mathrm{Hom}_R(M_i, M_j)$. It suffices to show that $(N_i)s_{ij} \subseteq N_j$ for each i and j with $1 \leqslant i < j \leqslant k$. Since $c(M_i) \geqslant c(M_j)$ and $M_i \not\cong M_j$, we have $c(\mathrm{Ker}(s_{ij})) \geqslant 1$. Hence

$$c((N_i)s_{ij}) \leqslant c(N_i) - 1 \leqslant c(N_1) - 1 \leqslant c(N_j)$$

and it follows that $(N_i)s_{ij} \subseteq N_j$.

THEOREM 15. *Every module over a serial ring is exact.*

Proof. Let R be a serial ring with $J = J(R)$ and ${}_RM$ be a left R-module.

Since there are only finitely many non-isomorphic uniserial modules, using Lemma 2, we may assume

$$_RM = \bigoplus_{i=1}^{n} M_i,$$

where each $_RM_i$ is uniserial, $c(M_1) = c_1 \geqslant \cdots \geqslant c(M_n) = c_n$, and $_RM_i \not\cong {_RM_j}$ whenever $i \neq j$. Suppose that

$$\begin{aligned} l_1 = c(M_1) = \cdots = c(M_{k_1}) > l_2 = c(M_{k_1+1}) = \cdots = c(M_{k_2}) \\ > \cdots > l_t = c(M_{k_{t-1}+1}) = \cdots = c(M_{k_t}), \end{aligned}$$

where $k_t = n$. Let $S_i = \mathrm{End}(_RM_i)$ and $S = \mathrm{End}(_RM)$. The exact bimodule $_R(M_i)_{S_i}$ has the unique composition series

$$M_i > JM_i > \cdots > J^{c_i-1}M_i > 0.$$

By Lemma 14, $_RM_S$ has a composition series

$$\begin{aligned} M &= M_1 \oplus \cdots \oplus M_{k_1} \oplus M_{k_1+1} \oplus \cdots \oplus M_n \\ &> M_1 \oplus \cdots \oplus M_{k_1-1} \oplus JM_{k_1} \oplus M_{k_1+1} \oplus \cdots \oplus M_n \\ &> M_1 \oplus \cdots \oplus M_{k_1-2} \oplus JM_{k_1-1} \oplus JM_{k_1} \oplus M_{k_1+1} \oplus \cdots \oplus M_n \\ &> \cdots \quad \cdots \\ &> JM_1 \oplus \cdots \oplus JM_{k_1} \oplus M_{k_1+1} \oplus \cdots \oplus M_n \\ &> JM_1 \oplus \cdots \oplus JM_{k_1-1} \oplus J^2M_{k_1} \oplus M_{k_1+1} \oplus \cdots \oplus M_n \\ &> \cdots \quad \cdots \\ &> J^{l_1-l_2}M_1 \oplus \cdots \oplus J^{l_1-l_2}M_{k_1} \oplus M_{k_1+1} \oplus \cdots \oplus M_n \\ &> J^{l_1-l_2}M_1 \oplus \cdots \oplus J^{l_1-l_2}M_{k_1} \oplus M_{k_1+1} \oplus \cdots \\ &\quad \oplus M_{k_2-1} \oplus JM_{k_2} \oplus M_{k_2+1} \oplus \cdots \oplus M_n \\ &> \cdots \quad \cdots \\ &> J^{l_1-l_2+1}M_1 \oplus \cdots \oplus J^{l_1-l_2+1}M_{k_1} \oplus JM_{k_1+1} \oplus \cdots \\ &\quad \oplus JM_{k_2} \oplus M_{k_2+1} \oplus \cdots \oplus M_n \\ &> \cdots \quad \cdots \\ &> J^{l_1-1}M_1 \oplus \cdots \oplus J^{l_1-1}M_{k_1} \oplus J^{l_2-1}M_{k_1+1} \oplus \cdots \oplus J^{l_t-1}M_n \\ &> J^{l_1-1}M_1 \oplus \cdots \oplus J^{l_t-1}M_{n-1} \oplus 0 \\ &> \cdots \quad \cdots > 0. \end{aligned}$$

Each composition factor of ${}_RM_S$ is of the form

$$F = \frac{\cdots \oplus J^iM_j \oplus \cdots}{\cdots \oplus J^{i+1}M_j \oplus \cdots} \overset{f}{\cong} {}_R(J^iM_j/J^{i+1}M_j)$$

which is a balanced $R-S_j$-bimodule since it is a composition factor of exact bimodule ${}_R(M_j)_{S_j}$. Hence ${}_RF$ is balanced. Let $g \in \operatorname{End}({}_RF)$. Then $f(g)$ is given by right multiplication of an element $s_j \in S_j$, which implies that g is given by right multiplication of

$$\begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & s_j & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} \in S.$$

Hence ${}_RF_S$ is balanced and so ${}_RM$ is exact by Lemma 1. Similarly each right R-module is exact.

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